

ACCELERATION CONTROL IN NONLINEAR VIBRATING SYSTEMS BASED ON DAMPED LEAST SQUARES

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Abstract

A discrete time control algorithm using the damped least squares is introduced for acceleration and energy exchange controls in nonlinear vibrating systems. It is shown that the damping constant of least squares and sampling time step of the controller must be inversely related to insure that vanishing the time step has little effect on the results. The algorithm is illustrated on two linearly coupled Duffing oscillators near the 1:1 internal resonance. In particular, it is shown that varying the dissipation ratio of one of the two oscillators can significantly suppress the nonlinear beat phenomenon.

1 Introduction

The damped least squares is a simple but effective analytical manipulation that helps to avoid singularity in practical minimization and control algorithms. It is also known as Levenberg-Marquardt method [11]. In order to illustrate the idea in simple terms, let us consider the minimization problem

$$\|E - A\delta u\|^2 \rightarrow \min \quad (1)$$

where $E \in R^n$ is a given vector, the notation $\|\dots\|$ indicates the Euclidean norm in R^n , A is typically a Jacobian matrix of n rows and m columns, and $\delta u \in R^m$ is an unknown minimization vector. Although a formal solution of this problem is given by $\delta u = (A^T A)^{-1} A^T E$, the matrix product $A^T A$ may appear to be singular so that no unique solution is possible. This fact usually points to multiple possibilities of achieving the same result unless specific conditions are imposed on the vector δu . The idea of damped least squares is to avoid such conditioning by adding one more quadratic form to the left hand side of expression (1) as follows

$$\|E - A\delta u\|^2 + \lambda \|\delta u\|^2 \rightarrow \min \quad (2)$$

where λ is a positive scalar number, which is often called *damping constant*; note that the term ‘damping’ has no relation to the physical damping or energy dissipation effects in vibrating systems usually characterized by *damping ratios*.

Now the inverse matrix includes the damping constant λ which can provide the uniqueness of solution given by

$$\delta u = (A^T A + \lambda I)^{-1} A^T E \quad (3)$$

where I is $n \times n$ identity matrix.

Different arguments are discussed in the literature regarding the use of damped least squares and best choice for the damping parameter λ [1], [2], [3], [4], [6], [7], [9], [10], [15], [16], [17], [23], [24]. In particular, it was noticed that the parameter λ may affect convergence properties of the corresponding algorithms. The parameter λ can be used also for other reason such as shifting the solution δu into desired area in R^m . In this case, the meaning of λ is rather close to that of Lagrangian multiplier imposing constraints on control inputs.

In case of dynamical systems, when all the quantities in (2) may depend on time, a continuous time analogue of (2) can be written in the integral form

$$\min_{\delta u} \int_0^T (\|E - A\delta u\|^2 + \lambda \|\delta u\|^2) dt \quad (4)$$

where the interval of integration is manipulated as needed, for instance, T can be equal to sampling time of the controller [12].

However, in the present work, a discrete time algorithm based on the damped least squares solution (3), which is used locally at every sample time t_n , is introduced. Such algorithm appears to be essentially discrete namely using different time step h may lead to different results. Nevertheless, if the parameters λ and h are coupled by some condition then the control input and system response show no significant dependence on the time step.

A motivation for the present work is as follows. In order to comply with the standard tool of dynamical systems dealing with differential equations, the methods of control are often formulated in continuous time by silently assuming that a discrete time analogous is easy to obtain one way or another whenever it is needed for practical reasons. For instance, data acquisition cards and on-board computers of ground vehicles usually acquire and process data once per 0.01 sec. Typically, based on the information, which is known about the system dynamic states and control inputs by the time instance t_n , the computer must calculate control adjustments for the next active time instance, t_{n+1} . The corresponding computational time should not therefore exceed $t_{n+1} - t_n = 0.01$ sec. Generally speaking, it is possible to memorize snapshots of the dynamic states and control inputs at some of the previous times $\{..., t_{n-2}, t_{n-1}\}$. However, increasing the volume of input data may complicate the code and, as a result, slow down the calculation process. Therefore, let us assume that updates for the control inputs are obtained by processing the system states, controls, and target states given only at the current time instance, t_n . The corresponding algorithm can be built on the system model described by its differential equations of motion and some

rule for minimizing the deviation (error) of the current dynamic states from the target. Recall that, in the present work, such a rule will be defined according to the damped least squares (2). Illustrating physical example of two linearly coupled Duffing oscillators is considered. It is shown that the corresponding algorithm, which is naturally designed and effectively working in discrete time, may face a problem of transition to the continuous time limit.

2 Problem formulation

Consider the dynamical system

$$\ddot{x} = f(x, \dot{x}, t, u) \quad (5)$$

where $x = x(t) \in R^n$ is the system position (configuration) vector, the overdot indicates derivative with respect to time t , the right-hand side $f \in R^n$ represents a vector-function that may be interpreted as a force per unit mass of the system, and $u = u(t) \in R^m$ is a control vector, whose dimension may differ from that of the positional vector x so that generally $n \neq m$.

In common words, the purpose of control $u(t)$ is to keep the acceleration $\ddot{x}(t)$ of system (5) as close as possible to the target $\ddot{x}^*(t)$. The term ‘close’ will be interpreted below through a specifically designed target function of the following error vector

$$E(t) = \ddot{x}^*(t) - \ddot{x}(t) \quad (6)$$

As discussed in Introduction, for practical implementations, the problem must be formulated in terms of the discrete time $\{t_k\}$ as follows. Let $x_k = x(t_k)$, $\dot{x}_k = \dot{x}(t_k)$, and $u_k = u(t_k)$ are observed at some time instance t_k . The corresponding target acceleration, $\ddot{x}_k^* = \ddot{x}^*(t_k)$, is assumed to be known. Then, taking into account (5) and (6), gives the following error at the same time instance

$$E_k = \ddot{x}_k^* - f(x_k, \dot{x}_k, t_k, u_k) \quad (7)$$

Now the purpose of control is to minimize the following target function

$$\begin{aligned} P_k &= \frac{1}{2} E_k^T W_k E_k \\ &= \frac{1}{2} [\ddot{x}_k^* - f(x_k, \dot{x}_k, t_k, u_k)]^T W_k [\ddot{x}_k^* - f(x_k, \dot{x}_k, t_k, u_k)] \end{aligned} \quad (8)$$

where W_k is $n \times n$ diagonal weight matrix whose elements are positive or at least non-negative functions of the system states, $W_k = W(x_k, \dot{x}_k, t_k)$.

Note that all the quantities in expression (8) represent a snapshot of the system at $t = t_k$ while including no data from the previous time step t_{k-1} . Since the control vector u_k cannot be already changed at time t_k then quantity P_k is out of control at time t_k . In other words expression (8) summarizes all what is observed *now*, at the time instance t_k . The question is how to adjust the control vector u for the next step t_{k+1} based on the information included in

(8) while the system state at $t = t_{k+1}$ is yet unknown, and no information from the previous times $\{\dots, t_{n-2}, t_{n-1}\}$ is available.

Let us represent such an update for the control vector in the form

$$u_{k+1} = u_k + \delta u_k \quad (9)$$

where δu_k is an unknown adjustment of the control input.

Replacing u_k in (8) by (9) and taking into account that

$$\begin{aligned} f(x_k, \dot{x}_k, t_k, u_{k+1}) &= f(x_k, \dot{x}_k, t_k, u_k) + A_k \delta u_k + O(\|\delta u_k\|^2) \\ A_k &= \partial f(x_k, \dot{x}_k, t_k, u_k) / \partial u_k \end{aligned} \quad (10)$$

gives

$$P_k = \frac{1}{2} (E_k - A_k \delta u_k)^T W_k (E_k - A_k \delta u_k) \quad (11)$$

where A_k is the Jacobian matrix of n rows and m columns.

Although the replacement u_k by u_{k+1} in (10) may look artificial, this is how the update rule for the control vector u is actually defined here. Namely, if u_k did not provide a minimum for $P_k(\ddot{x}_k^*, x_k, \dot{x}_k, t_k, u_k)$, then let us minimize $P_k(\ddot{x}_k^*, x_k, \dot{x}_k, t_k, u_k + \delta u_k)$ with respect to δu_k and then apply the adjusted vector (9) at least the next next time, t_{n+1} . Assuming that the variation δu_k is small, in other words, u_k is still close enough to the minimum, expansion (10) is applied. Now the problem is formulated as a minimization of the quadratic form (11) with respect to the adjustment δu_k . However, what often happens practically is that function (11) has no unique minimum so that equation

$$\frac{dP_k}{d\delta u_k} = 0 \quad (12)$$

has no unique solution. In addition, even if the unique solution does exist, it may not satisfy some conditions imposed on the control input due to the physical specifics of actuators. As a result, some constraint conditions may appear to be necessary to impose on the variation of control adjustment, δu_k . However, the presence of constraints would drastically complicate the problem. Instead, the target function (11) can be modified in order to move solution δu_k into the allowed domain. For that reason, let us generalize function (11) as

$$\begin{aligned} P_k &= \frac{1}{2} (E_k - A_k \delta u_k)^T W_k (E_k - A_k \delta u_k) \\ &\quad + \frac{1}{2} (B_k + C_k \delta u_k)^T \Lambda_k (B_k + C_k \delta u_k) \end{aligned} \quad (13)$$

where $\Lambda_k = \Lambda(x_k, \dot{x}_k, t_k)$ is a diagonal regularization matrix, $B_k = B(x_k, \dot{x}_k, t_k)$ is a vector-function of n elements, and $C_k = C(x_k, \dot{x}_k, t_k)$ is a matrix of n rows and m columns.

Note that the structure of new function (13) is a generalization of (2). Substituting (13) in (12), gives a linear set of equations in the matrix form whose solution δu_k brings relationship (9) to the form

$$u_{k+1} = u_k + (A_k^T W_k A_k + C_k^T \Lambda_k C_k)^{-1} (A_k^T W_k E_k - C_k^T \Lambda_k B_k) \quad (14)$$

The entire discrete time system is obtained by adding a discrete version of the dynamical system (5) to (14). Assuming that the time step is fixed, $t_{k+1} - t_k = h$, a simple discrete version can be obtained by means of Euler explicit scheme as follows

$$\begin{aligned} x_{k+1} &= x_k + h v_k \\ v_{k+1} &= v_k + h f(x_k, v_k, t_k, u_k) \end{aligned} \quad (15)$$

Finally, equations (14) and (15) represent a discrete time dynamical system, whose motion should follow the target acceleration $\ddot{x}_k^* = \ddot{x}^*(t_k)$.

It will be shown in the next section that the structure of equation (14) does not allow for the transition to continuous limit of the entire dynamic system (14) through (15), unless some specific assumption are imposed on the parameters in order to guarantee that $\delta u_k = O(h)$ as $h \rightarrow 0$.

3 The illustrating example

The algorithm, which is designed in the previous section, is applied now to a two-degrees-of-freedom nonlinear vibrating system for an *active* control of the energy exchange (nonlinear beats) between the two oscillators. The problem of *passive* control of energy flows in vibrating systems is of great interest [22], and it is actively discussed from the standpoint of nonlinear beat phenomena [14]. The beating phenomenon takes place when frequencies of the corresponding linear oscillators are either equal or at least close enough to each other.

For illustrating purposes, let us consider two unit-mass Duffing oscillators of the same linear stiffness K coupled by the linear spring of stiffness γ . The system position is described by the vector-function of coordinates, $x(t) = (x_1(t), x_2(t))^T$. Introducing the parameters $\Omega = (\gamma + K)^{1/2}$ and $\varepsilon = \gamma/(\gamma + K)$, brings the differential equations of motion to the form

$$\begin{aligned} \dot{x}_1 &= v_1 \\ \dot{x}_2 &= v_2 \\ \dot{v}_1 &= -2\zeta\Omega v_1 - \Omega^2 x_1 + \varepsilon(\Omega^2 x_2 - \alpha x_1^3) \equiv f_1(x_1, x_2, v_1) \\ \dot{v}_2 &= -2u\Omega v_2 - \Omega^2 x_2 + \varepsilon(\Omega^2 x_1 - \alpha x_2^3) \equiv f_2(x_1, x_2, v_2, u) \end{aligned} \quad (16)$$

where α is a positive parameter, ζ and u are damping ratios¹ of the first and the second oscillators, respectively; the damping ratio u , which is explicitly shown as an argument of the function $f_2(x_1, x_2, v_2, u)$, will be considered as a control input.

¹As mentioned in Introduction, the damping (dissipation) ratio should not be confused with the damping coefficient λ .

The problem now is to find such variable damping ratio $u = u(t)$ under which the second oscillator accelerates as close as possible to the given (target) acceleration, $\ddot{x}_2^*(t)$.

Following the discussion of the previous section, let us consider the problem in the discrete time $\{t_k\}$. In order to avoid confusion, the iterator k will be separated from the vector component indexes by coma, for instance, $x_k = (x_{1,k}, x_{2,k})^T$. Since only the second mass acceleration is of interest and the system under consideration includes only one control input u , then, assuming the weights to be constant, gives

$$W_k = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_k = \frac{\partial}{\partial u_k} \begin{bmatrix} f_{1,k} \\ f_{2,k} \end{bmatrix}$$

where $f_{1,k} \equiv f_1(x_{1,k}, x_{2,k}, v_{1,k})$ and $f_{2,k} \equiv f_2(x_{1,k}, x_{2,k}, v_{2,k}, u_k)$, and other matrix terms become scalar quantities, say, $\Lambda_k = \lambda$, $B_k = b$, and $C_k = 1$. The unities in W_k and C_k can always be achieved by re-scaling the target function and parameters λ and b . Note that re-scaling the target function by a constant factor has no effect on the solution of equation (12).

As a result, the target function (13) takes the form

$$P_k = \frac{1}{2} \left(\ddot{x}_{2,k}^* - f_{2,k} - \frac{\partial f_{2,k}}{\partial u_k} \delta u_k \right)^2 + \frac{\lambda}{2} (b + \delta u_k)^2 \quad (17)$$

In this case, equation (12) represents a single linear equation with respect to the scalar control adjustment, δu_k . Substituting the corresponding solution in (14) and taking into account (15), gives the discrete time dynamical system

$$u_{k+1} = u_k - \frac{(f_{2,k} - \ddot{x}_{2,k}^*)(\partial f_{2,k}/\partial u_k) + \lambda b}{(\partial f_{2,k}/\partial u_k)^2 + \lambda} \quad (18)$$

and

$$\begin{aligned} x_{1,k+1} &= x_{1,k} + h v_{1,k} \\ x_{2,k+1} &= x_{2,k} + h v_{2,k} \\ v_{1,k+1} &= v_{1,k} + h f_{1,k} \\ v_{2,k+1} &= v_{2,k} + h f_{2,k} \end{aligned} \quad (19)$$

Let us assume now that the target acceleration \ddot{x}_2^* is zero, in other words, the purpose of control is to minimize acceleration of the second oscillator at any sample time t_k as much as possible. Let us set still arbitrary parameter b also to zero. Then the target function (17) and dynamical system (18) and (19) take the form

$$P_k = \frac{1}{2} \left[f_2(x_{1,k}, x_{2,k}, v_{2,k}, u_k) + \frac{\partial f_2(x_{1,k}, x_{2,k}, v_{2,k}, u_k)}{\partial u_k} \delta u_k \right]^2 + \frac{\lambda}{2} (\delta u_k)^2 \quad (20)$$

$$\begin{aligned}
u_{k+1} &= u_k + \frac{2\Omega v_{2,k}}{4\Omega^2 v_{2,k}^2 + \lambda} f_2(x_{1,k}, x_{2,k}, v_{2,k}, u_k) \\
x_{1,k+1} &= x_{1,k} + h v_{1,k} \\
x_{2,k+1} &= x_{2,k} + h v_{2,k} \\
v_{1,k+1} &= v_{1,k} + h f_1(x_{1,k}, x_{2,k}, v_{1,k}) \\
v_{2,k+1} &= v_{2,k} + h f_2(x_{1,k}, x_{2,k}, v_{2,k}, u_k)
\end{aligned} \tag{21}$$

where the functions f_1 and f_2 are defined in (16).

As follows from the first equation in (21), transition to the continuous time limit for the entire system (21) would be possible under the condition that

$$\frac{2\Omega v_{2,k}}{4\Omega^2 v_{2,k}^2 + \lambda} = O(h), \quad \text{as } h \rightarrow 0 \tag{22}$$

Condition (22) can be satisfied by assuming that $\Omega = O(h)$. Such an assumption, however, makes little if any physical sense. As an alternative choice, the condition $\lambda = O(h^{-1})$ can be imposed by setting, for instance,

$$\lambda h = \lambda_0 \tag{23}$$

where λ_0 remains finite as $h \rightarrow 0$.

However, condition (23) essentially shifts the weight on control to the second term of the target function (17) so that the function asymptotically takes the form

$$P_k \simeq \frac{\lambda_0}{2h} (\delta u_k)^2, \quad \text{as } h \rightarrow 0 \tag{24}$$

Such a target function leads to the solution $\delta u_k = 0$, which effectively eliminates the control equation. In other words, the iterative algorithm seems to be essentially discrete. As a result, the control input u_k , generated by the first equation in (21), depends upon sampling time interval h . Let us illustrate this observation by implementing the iterations (21) under the fixed set of parameters, $\varepsilon = 0.1$, $\Omega = 1.0$, $\alpha = 1.5$, $\zeta = 0.025$, and initial conditions, $u_0 = 0.025$, $x_{1,0} = 1.0$, $x_{2,0} = 0.1$, $v_{1,0} = v_{2,0} = 0$. The values to vary are two different sampling time intervals, $h = 0.01$ and $h = 0.001$, and three different values of the damping constant, $\lambda = 0.1$, $\lambda = 1.0$, and $\lambda = 10.0$. For comparison reason, Fig. 1 shows time histories of the system coordinates under the fixed control variable $u = \zeta$. This (no control) case corresponds to free vibrations of the model (16) whose dynamics represent a typical beat-wise decaying energy exchange between the two oscillators. As mentioned at the beginning of this section, the beats are due to the 1:1 resonance in the generating system ($\varepsilon = 0$, $u = \zeta = 0$); more details on non-linear features of this phenomenon, the related analytical tools, and literature overview can be found in [20] and [14]. In particular, the standard averaging method was applied to the no damping case of system (16) in [20].

Now the problem is to suppress the beat phenomenon by preventing the energy flow from the first oscillator into the second oscillator. As follows from Figs. 2 through 5, such a goal can be achieved by varying the damping ratio of the second oscillator, $\{u_k\}$, during the vibration process according to the algorithm² (21). First, the diagrams in Figs. 2 and 3 confirm that the sampling time interval h represents an essential parameter of the entire control loop. In particular, decreasing the sampling interval from $h = 0.01$ to $h = 0.001$ effectively increases the strength of the control; compare fragments (b) in Figs. 2 and 3. However, if such decrease of the sampling time is accompanied by the increase of λ according to condition (23), then the strength of control remains practically unchanged; compare now fragments (b) in Figs. 2 and 4. As follows from fragments (a) in Figs. 2 and 4, the above modification of both parameters, h and λ , also brings some difference in the system response during the interval $80 < t < 150$, but this is rather due to numerical effect of the time step. Finally, analyzing the diagrams in Figs. 3 and 5, shows that reducing the parameter λ as many as ten times under the fixed time step h leads to a significant increase of the control input $\{u_k\}$ with a minor effect on the system response though. Therefore the parameter λ can be used for the purpose of satisfying some constraint conditions on the control inputs $\{u_k\}$ in case such conditions are due to physical limits of the corresponding actuators. In addition, let us show that parameter λ may affect the convergence of algorithm (21) based on the following convergence criterion [18]:

For a fixed point z_ to be a point of attraction of the algorithm $z_{k+1} = G(z_k)$ a sufficient condition is that the Jacobian matrix of G at the point z_* has all its eigenvalues numerically less than 1, and a necessary condition is that they are numerically at most 1. The geometric rate of convergence is the numerically largest eigenvalue of this Jacobian.*

Applying this criterion to the algorithm (21) at zero point, gives that one of the eigenvalues is always zero, $q_0 = 0$, whereas another four eigenvalues, q_i ($i = 1, \dots, 4$) are proportional to the time step, $q_i = hp_i$, where the coefficients p_i are given by the roots of algebraic equation

$$p^4 + 2\zeta\Omega p^3 + 2\Omega^2 p^2 + 2\zeta\Omega^3 p + (1 - \varepsilon^2)\Omega^4 = 0 \quad (25)$$

As follows from (25), the damping coefficient λ has no influence on the convergence condition near the equilibrium point, and the convergence can always be achieved under a small enough time step h . Nevertheless, the damping coefficient may appear to affect the convergence away from the equilibrium point. In this case, analytical estimates for eigen values of the Jacobian become technically complicated unless $\varepsilon = 0$, when four of the five eigenvalues vanish as $h \rightarrow 0$, except one eigenvalue, which is estimated by

$$q = - \left(1 + \frac{\lambda}{4\Omega^2 v_2^2} \right)^{-1} \quad (26)$$

²Note that, although the algorithm is designed to suppress accelerations of the second oscillator, acceleration and energy levels of vibrating systems are related.

This root gives $q \rightarrow q_0 = 0$ as $v_2 \rightarrow 0$. However, when $v_2 \neq 0$, equation (26) gives the estimate $0 < q \leq 1$ as $\infty > \lambda \geq 0$. Therefore, only the necessary convergence condition is satisfied for $\lambda = 0$.

4 Conclusions

In this work, a discrete time control algorithm for nonlinear vibrating systems using the damped least squares is introduced. It is shown that the corresponding damping constant λ and sampling time step h must be coupled by the condition $\lambda h = \text{constant}$ in order to preserve the result of calculation when varying the time step. In particular, the above condition prohibits a direct transition to the continuous time limit. This conclusion and other specifics of the algorithm are illustrated on the nonlinear two-degrees-of-freedom vibrating system in the neighborhood of 1:1 resonance. It is shown that the dissipation ratio of one of the two oscillators can be controlled in such way that prevents the energy exchange (beats) between the oscillators. From practical standpoint, controlling the dissipation ratio can be implemented by using devices based on the physical properties of magnetorheological fluids (MRF) [8], [19]. In particular, different MRF dampers are suggested to use for semi-active ride controls of ground vehicles and seismic response reduction.

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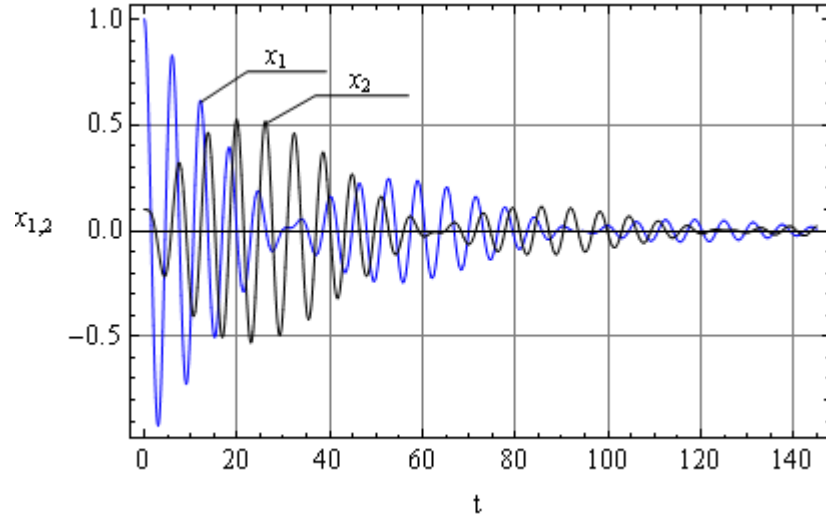


Figure 1: No control beat dynamics with the decaying energy exchange between two Duffing's oscillators; $u = \zeta = 0.025$.

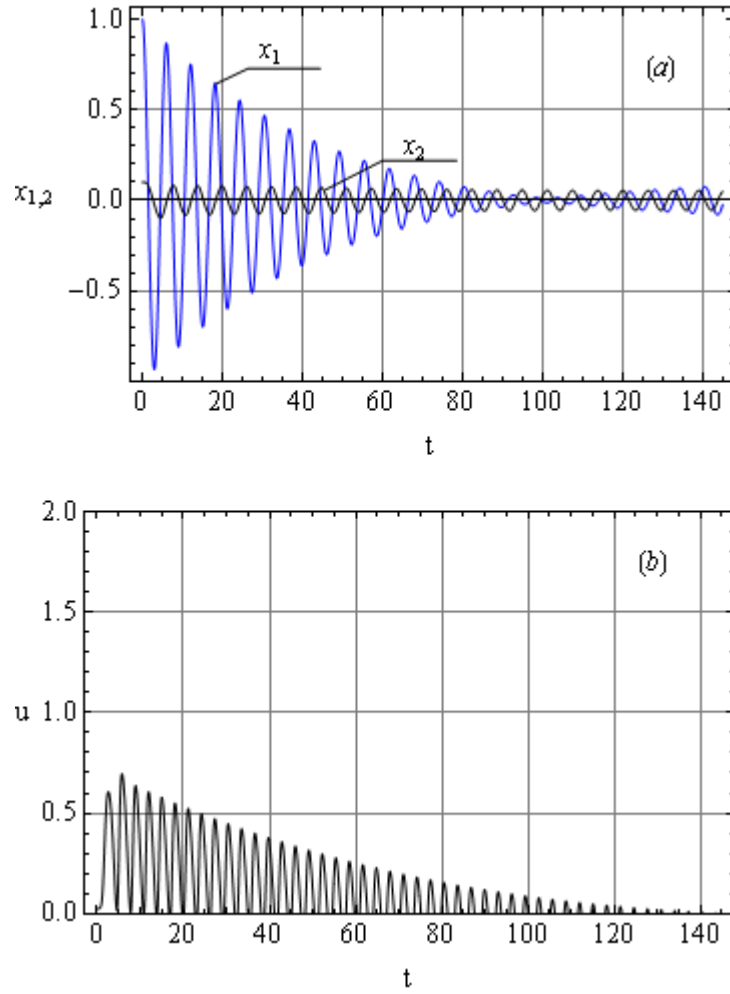


Figure 2: Beat suppression under the time increment $h = 0.01$ and weight parameter $\lambda = 1.0$: (a) the system response, (b) control input - the damping ratio of second oscillator.

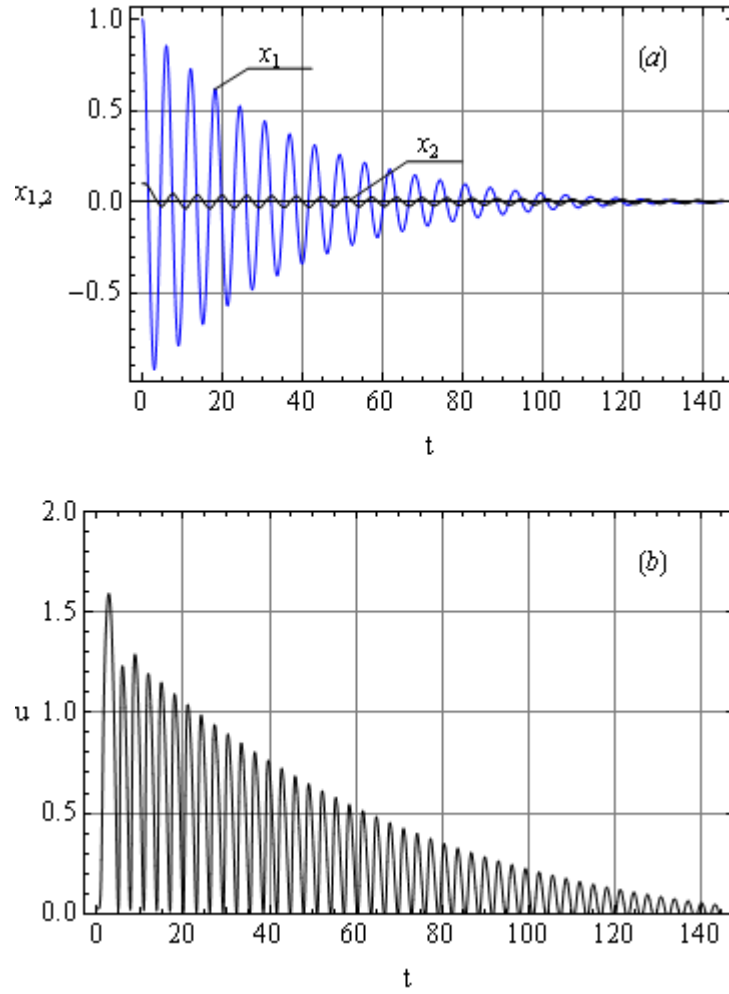


Figure 3: Beat suppression under the reduced time increment $h = 0.001$ and the same weight parameter $\lambda = 1.0$: (a) the system response, (b) control input - the damping ratio of second oscillator.

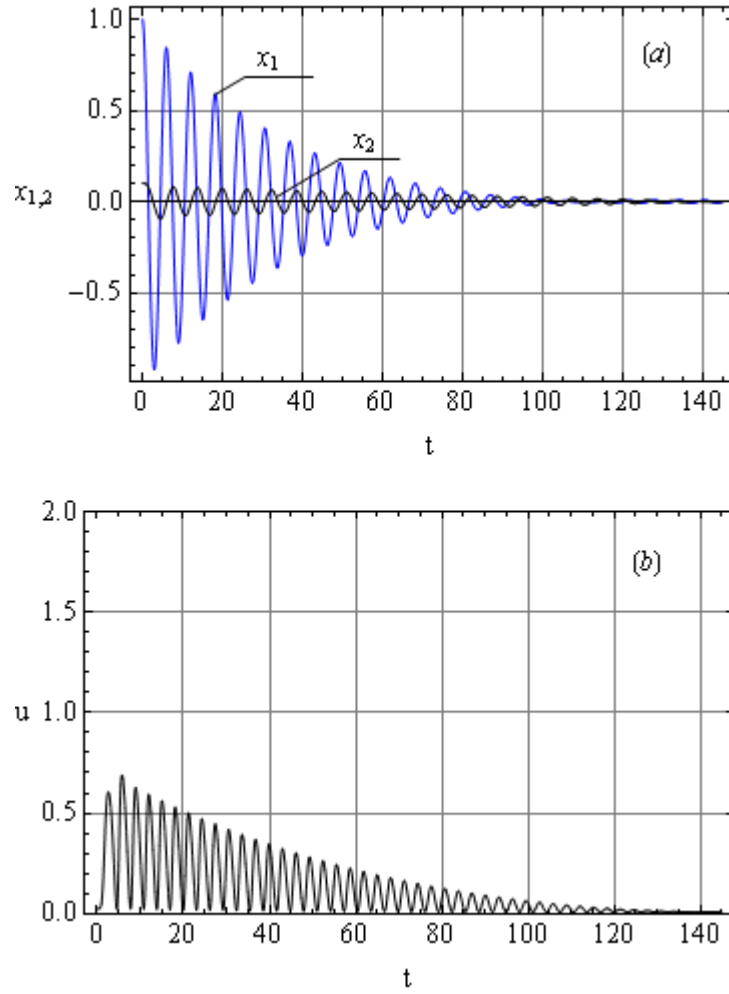


Figure 4: Beat suppression under the reduced time increment $h = 0.001$ but increased weight parameter $\lambda = 10.0$: (a) the system response, (b) control input - the damping ratio of second oscillator.

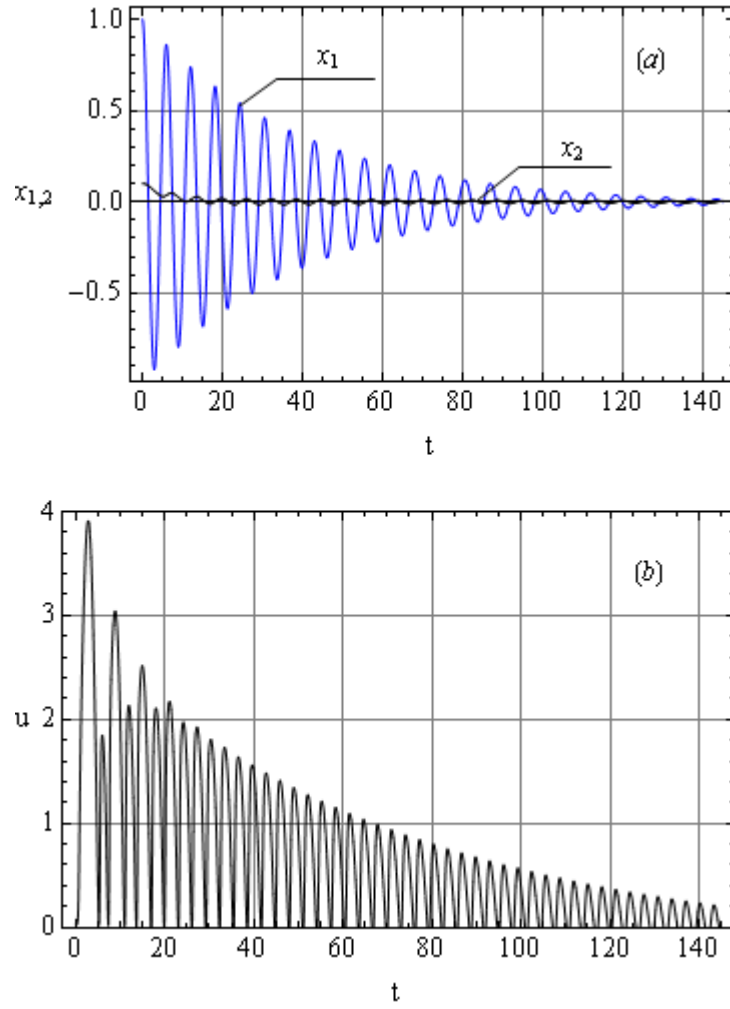


Figure 5: Beat suppression under the reduced time increment $h = 0.001$ and vanishing weight parameter $\lambda = 0.1$: (a) the system response, (b) control input - the damping ratio of second oscillator.